Geometry and Topology
For morning of October 25th

Problem 1 *Consider the mean curvatur flow (MCF for short) $f : M \times I \to \mathbb{R}^n$ of a hypersurface M in $\mathbb{R}^n$ where $f_t = f(\cdot, t) : M \to \mathbb{R}^n$. This means*

$$\left(\frac{\partial f}{\partial t}\right)^{\perp} = H(t)$$

where $^{\perp}$ denotes the normal component, and $H(t)$ is the mean curvature vector of $f_t(M)$. In particular when $\frac{\partial f}{\partial t} = T$, a constant vector, M_t is called a translating soliton, which is just a parallel transport of M in the direction T.

(1) When $n = 2$, show that the MCF of the grim reaper γ in $\mathbb{R}^2$

$$y = -\log \cos x$$

is the translating soliton with $T = (0, 1)$.

(2) Show that $M = \gamma \times \mathbb{R}^k$ is a translating soliton in $\mathbb{R}^{k+2}$.

Problem 2 *Let (M^n, g) be a closed, orientable n-dimensional Riemannian manifold with positive Ricci curvature.*

(a) Prove that the first Betti number b_1 of M vanishes.

(b) Suppose $Ric_M \geq (n-1)\kappa > 0$, show that $\lambda_1 \geq n\kappa$, where λ_1 is the first eigenvalue of the Laplace-Beltrami operator Δ on (M, g).

Problem 3 *Let I be the interval $[0, 1]$. For a topological space B, say homeomorphisms $g_0, g_1 : B \to B$ are isotopic if they are homotopic via a homotopy $G : B \times I \to B$ with each $G_t : B \to B$ defined by $G_t(b) = G(b, t)$ also a homeomorphism.*

(1) Show that any orientation-preserving homeomorphism $f : D^2 \to D^2$ of the closed unit disc is isotopic to a homeomorphism which is the identity on the boundary S^1.

(You can use the fact that any orientation-preserving homeomorphim $\phi : S^1 \to S^1$ is isotopic to the identity).

(2) Show that a homeomorphism $f : D^2 \to D^2$ of the unit disc is isotopic to the identity or the reflection along the x-axis.